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Numerical evidence of the critical percolation probability $P_c = 1$ for site problems on Sierpinski gaskets†

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Abstract. In this paper we propose a new method, the alternating translational-dilation method (ATDM), and use it in combination with the Monte Carlo technique to calculate the critical percolation probability for Sierpinski gaskets (SG). We obtain numerical evidence of the critical percolation probability $P_c = 1$ for site problems on SG and an approximate relation $P_c = 1 - 1/N$ in two-dimensions, where N is the number of stages of SG.

1. Introduction

In recent years, fractal structures, both deterministic (self-similar) like the SG [1-3] and stochastic (statistically self-similar) like the percolation clusters [1, 4] at criticality, have been received much attention. SG lattices, consisting of sites connected by bonds, are valuable model systems for many theoretical purposes, including the study of phase transitions and the modelling of transport phenomena.

In the study of fractals, SG embedded in d -dimensional Euclidean spaces are well suited both for analytical considerations [1, 3, 5-7] and for numerical simulations [8, 9]. We note that the SG result from this construction as the special case $b = 2$ (b is the number of layers of a smaller generator (a basic geometrical unit)) of Sierpinski-type fractals. In figure 1 we display Sierpinski-type fractal units in two dimensions ($d = 2$) for the cases $b = 2, 3, 5$ and 10. Sierpinski-type fractal lattices are built up recursively by the ATDM (see § 2) presented in this paper.

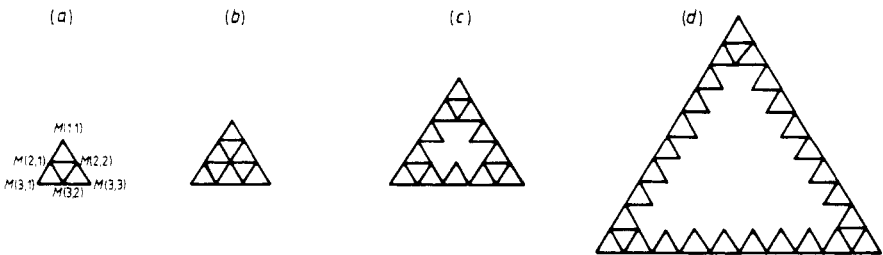


Figure 1. Sierpinski-type fractal generators (basic geometrical units) for different b values in two dimensions. (a) $b = 2$; (b) 3; (c) 5; (d) 10.

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In this paper we focus our attention on the SG. We illustrate their construction method by both formulae and figures, and use the method of construction, as well as the Monte Carlo technique, to compute the critical percolation probability for site problems on SG.

The SG lattices have a finite order of ramification and neither finite-temperature phase transitions [5, 10] nor $P_c = 1$ rigorously. Our results will support this conclusion.

2. The alternating translational-dilation method of construction for Sierpinski-type fractal lattices

The Sierpinski-type fractals form a family of self-similar structures, which can be built in any Euclidean dimension d (see figure 1). For the convenience of computing P_c on SG numerically, one needs to transfer the geometry problem for SG lattices into a digital problem. In this paper, for Sierpinski-type fractal lattices as shown in figure 1 we propose a transformation method, the alternating translational-dilation method (ATDM). We deduce the computational formulae for the ATDM as follows.

We first assign different initial values for sites of the smallest generator (a triangle), for example $M(1, 1) = 1, M(2, 1) = 7, M(2, 2) = 2$ (see figure 2(a)).

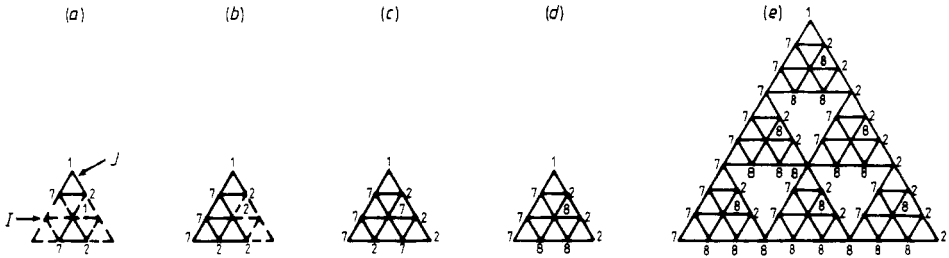


Figure 2. The construction of lattices for Sierpinski-type fractals.

Then the generator is translated and dilated towards the lower end; the related sites (see figure 2(a)) are assigned according to the equation

$$M(I, J) = M(I - (b - 1)b^{m-1}, J - b^{m-1}D) \tag{1}$$

where

$$1 \leq m \leq N \quad (b - 1)b^{m-1} + 1 \leq I \leq b^m + 1$$

$$Db^{m-1} + 1 \leq J \leq I - (b - 1 - D)b^{m-1}.$$

Here m is the serial number of translation; I and J represent the rows and columns respectively, $M(I, J)$ is an element of arrays and D is a dilation parameter, $1 \leq D \leq b - 2$. One should note, for $b = 2$, that (1) is eliminated.

When the generator is translated towards the lower left (see figure 2(b)), the values of related sites are given by

$$M(I, J) = M(I - T \times b^{m-1}, J) \tag{2}$$

where

$$1 \leq T \leq b - 1 \quad Tb^{m-1} + 2 \leq I \leq (T + 1)b^{m-1} + 1$$

$$1 \leq J \leq I - Tb^{m-1}$$

and T is a translation parameter.

Then the generator is translated towards the lower right (see figure 2(c)) and we have

$$M(I, J) = M(I - Tb^{m-1}, J - Tb^{m-1}) \tag{3}$$

where

$$Tb^{m-1} + 2 \leq I \leq (T + 1)b^{m-1} + 1 \quad Tb^{m-1} + 1 \leq J \leq I.$$

Besides the three basic formulae stated above, we also derive the following two formulae to reassign the sites taking account of the particular neighbours (three neighbour sites already investigated, see § 3) of the sites to be investigated.

For the central site (e.g., the site $M(3, 2)$) of $b = 2, 3, 5, 10$ in figure 1

$$M(I, J) = \text{a positive integer (e.g., 8)} \tag{4}$$

where $I = 2b^{m-1} + 1, J = b^{m-1} + 1$.

For the sites $M(4, 2), M(4, 3)$ for $b = 3$ and $M(6, 2), \dots, M(6, 5)$ for $b = 5$

$$M(I, J) = \text{a positive integer (e.g., 8)} \tag{5}$$

where $I = b^m + 1, J = Tb^{m-1} + 1$. The integer 8 must be equal to the value of equation (4) because they belong to the same category of sites (see § 3). One should also note, for $b = 2$, that (5) is eliminated.

For the cutouts of lattices, the computer can automatically assign zero to the cutout sites.

If $b = 2$ (for the SG), equations (2)-(4) will respectively become

$$M(I, J) = M(I - 2^{m-1}, J) \tag{2'}$$

where

$$2^{m-1} + 2 \leq I \leq 2^m + 1, 1 \leq J \leq I - 2^{m-1},$$

$$M(I, J) = M(I - 2^{m-1}, J - 2^{m-1}) \tag{3'}$$

where $2^{m-1} + 2 \leq I \leq 2^m + 1, 2^{m-1} + 1 \leq J \leq I$, and

$$M(I, J) = \text{a positive integer (e.g., 8)} \tag{4'}$$

where $I = 2^m + 1, J = 2^{m-1} + 1$.

As m is increased the translation and dilation of generators are alternately generated, and assignment processes for lattices are repeated by equations (1)-(5). Finally, the required lattice of Sierpinski-type fractal structures is formed.

Here is an example to illustrate the method (ATDM) of construction by applying it to a Sierpinski-type fractal structure ($b = 3, N = 2$).

According to (1), when $m = 1, D = 1, 3 \leq I \leq 4, 2 \leq J \leq I - 1$ (i.e. when $I = 3, J = 2; I = 4, J = 2, 3$) we have $M(3, 2) = M(1, 1) = 1$ and $M(4, 2) = M(2, 1) = 7, M(4, 3) = M(2, 2) = 2$ (see figure 2(a)).

From (2), $1 \leq T \leq 2$, when $T = 1, I = 3, J = 1, 2$, we get $M(3, 1) = M(2, 1) = 7, M(3, 2) = M(2, 2) = 2$. When $T = 2, I = 4, 1 \leq J \leq 2$ (or $J = 1, 2$), we have $M(4, 1) = M(2, 1) = 7, M(4, 2) = M(2, 2) = 2$. So the translation towards the lower left is completed (see figure 2(b)).

For (3), when $T = 1, I = 3, 2 \leq J \leq 3$, we obtain $M(3, 2) = M(2, 1) = 7, M(3, 3) = M(2, 2) = 2$. When $T = 2, I = 4, 3 \leq J \leq 4$, we gain $M(4, 3) = M(2, 1) = 7, M(4, 4) = M(2, 2) = 2$. Then the translation towards the lower right is completed (see figure 2(c)).

For $M(3, 2)$ and $M(4, 2)$, $M(4, 3)$, according to (4) and (5) we assign 8 to the sites; finally we obtain figure 2(d).

If $m = 2$, the calculation processes are repeated by (1)–(5) and we obtain figure 2(e).

We divide the sites on these lattices into four categories according to a consideration of generation and combination of clusters. The first category of sites is those whose values equal 1; the second equal 2; the third equal 8; and the fourth equal 7 (see figure 2(e)).

3. Site occupation and cluster calculation on sg

The sg lattice is formed by equations (2')–(4'). Sites on sg are randomly occupied according to rows (I) and columns (J). The probabilities P_i ($i = 1, 2, 3, \dots$) are compared with a pseudorandom number $Y(0-1)$. If $P_i < Y$, the site $M(I, J)$ is not occupied and is assigned as zero, otherwise a positive integer is assigned to the $M(I, J)$. If the site $M(I, J)$ is occupied, nearest-neighbouring sites already compared with Y are necessarily investigated to see whether they are occupied or not. If so, it belongs to the cluster composed of neighbouring sites and the serial number of the cluster is assigned to the site $M(I, J)$ and the size of the cluster is increased by unity. If not, the site $M(I, J)$ forms a new cluster (its size is 1), and a new number which is greater than the last one by 1 is assigned to the new cluster, and so on. Two clusters coalesce into a greater cluster if two clusters are joined by (at least) a bond joining two sites which belong to different clusters.

The first category of sites has no neighbouring site which has been earlier investigated; the second has two sites, $M(I, J-1)$ and $M(I-1, J-1)$; the third has three, $M(I, J-1)$, $M(I-1, J-1)$ and $M(I-1, J)$; the fourth has only one, $M(I-1, J)$.

If two clusters coalesce into a new greater cluster, the serial number (e.g., 3) of sites of the former cluster is changed to be the serial number (e.g., 5) of sites of the latter cluster. The size of the new cluster then is equal to the sum of the sizes of two clusters, and finally zero is assigned to the former cluster size.

4. Critical percolation probability P_c and discussions

We applied the modified second moment

$$M = \sum_i S_i^2 \left(\sum_i S_i \right)^{-2} \quad (6)$$

defined by Dean [11], here S_i ($i = 1, 2, \dots$) are the sizes of clusters and the summation runs over all the lattice clusters (including the largest cluster).

According to Dean's method, P_c is taken to be that value of P at which $\Delta M / \Delta P$ is a maximum, Δ indicating an increment of one step. Dean chose ΔM to be a constant value and found the value of P at which the corresponding increment ΔP was a minimum. And in the present work we choose ΔP to be a constant value (0.01) and set P_i ($i = 1, 2, \dots$), so $\Delta P = P_{i+1} - P_i$ and M_i ($i = 1, 2, \dots$) is calculated from (6). The critical percolation probability is given by

$$P_c = \max[(M_{i+1} - M_i) / \Delta P]. \quad (7)$$

Table 1. Critical percolation probabilities P_c on SG, σ_c is the standard deviation, σ_t is an error between the experimental value and that predicted from (8).

| Stage N | Experimental value | | Predicted value (8) | |
|-----------|--------------------|---------------|---------------------|------------|
| | P_c | $\pm\sigma_c$ | P_c | σ_t |
| 3 | 0.724 | 0.089 | 0.667 | -0.057 |
| 4 | 0.770 | 0.101 | 0.750 | -0.020 |
| 5 | 0.802 | 0.023 | 0.800 | -0.002 |
| 6 | 0.820 | 0.032 | 0.833 | +0.013 |
| 7 | 0.882 | 0.031 | 0.857 | -0.025 |
| 8 | 0.885 | 0.016 | 0.875 | -0.010 |

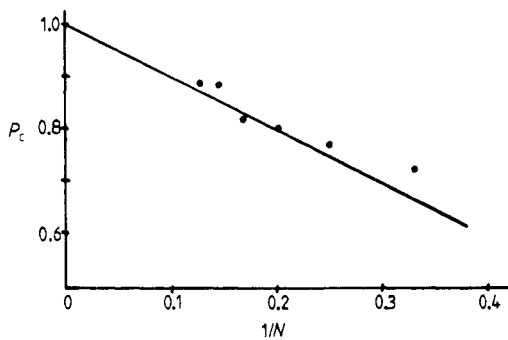


Figure 3. Plot of experimental values of P_c against $1/N$ on SG in two dimensions, compared with equation (8), $P_c = 1 - 1/N$, indicated by the straight line.

From the computation of the cluster-size distribution the critical percolation probability can be obtained from (7). The results for various stages of SG are summarised in table 1, where the averages are taken over five runs.

It can be seen from table 1 that the values of P_c are increased with the increase of stages N and the standard deviations σ_c in P_c are decreased with N although there are some small fluctuations in σ_c .

We also plot P_c against $1/N$ in figure 3 and extrapolate the results for finite N to the limit $N = \infty$, and find the intercept $P_c = 1$ at $N = \infty$. This supports the conclusion $P_c = 1$. Furthermore, the relation of P_c with $1/N$ can be approximated by

$$P_c = 1 - 1/N. \tag{8}$$

The errors between the experimental values and values predicted from (8) are well within σ_c (see table 1).

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